Lack of consensus in social systems

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We propose an exactly solvable model for the dynamics of voters in a two-party system. The opinion formation process is modeled on a random network of agents. The dynamical nature of interpersonal relations is also reflected in the model, as the connections in the network evolve with the dynamics of the voters. In the infinite time limit, an exact solution predicts the emergence of consensus, for arbitrary initial conditions. However, before consensus is reached, two different metastable states can persist for exponentially long times. One state reflects a perfect balancing of opinions, the other reflects a completely static situation. An estimate of the associated lifetimes suggests that lack of consensus is typical for large systems.

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Concepts and tools of modern nonequilibrium statistical physics lend themselves very directly to describing complex interacting systems, including phenomena which rely on human behavior, e.g. the emergence of collective organization in social systems. Recently variants of the voter model [1, 2, 3] have been used intensively to study collective phenomena, such as opinion formation or consensus creation [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Many of these efforts have focused on regular lattices [4, 5, 6, 7, 8, 9], which is justified in physical situations, but not in the context of the social sciences. In sociocultural situations, the interaction patterns between individuals find a better characterization as complex networks in which the connections or relationships (links) between individuals (nodes) can change in time. More precisely, the full dynamics of such a social network consists of (i) the opinion formation process taking place on the nodes, and (ii) the evolution of the underlying topological structure (links). The coupling between these two processes reflects how the connections of people influence their opinions, and how their opinions determine, in turn, their new connections. Although there is an increasing recent interest in modeling voter dynamics on graphs [10, 11], and on networks [12, 13, 14], the dynamics of the links (ii) is still disregarded in these studies. The coevolution of nodes *and* links – i.e., of the full network structure – has been studied only in a few works [15, 16, 17, 18, 19, 20].

In this letter, we investigate the voter dynamics on an *adaptive* disordered network characterized by (i)-(ii). In our network, each node j ("individual") carries a spin σ_j ("opinion") which can take two different values $\sigma_j = \pm 1$ [1]. At each time step, (i) the spins are updated random sequentially based on a simple majority rule: if they are connected to more positive than negative spins, their state will be positive in the next time step, and negative otherwise; in the case of a tie, the spin remains unchanged. Further, (ii) the links are updated as follows: two nodes carrying equal (unequal) spins are connected with probability p (q). In this letter, we focus on the special case $q = 1 - p$, leaving the general case to [21].

As an interpretation, we propose that this model mimics a two-party electoral system. During a campaign, the supporters of one party are keen to interact with supporters of the other party to try to change their opinion. This situation can be described by this model with $p < q$, when each agent has more interactions with opponents than with agents sharing the same opinion (according to the motto that *"convinced people do not need to be convinced again"*). On the other hand, when $p > q$, the agents tend to interact more with individuals sharing the same opinion (according to the motto *"united we are stronger"*). The latter behavior seems to be a simplified description of the process of political *polarization*, when all the members of a party agree with the official position of the party, as often occurs in post-election periods.

Our model is aimed to describe a free public debate in the sense that it does not consider the effects of central institutions or the mass media; neither lobbying, nor organized strategies (apart from possibly influencing the probability p) are taken into account. As a result, the model may also be appropriate to describe groups defined by criteria such as education, religion or ethnicity, rather than political opinion. Cultural assimilation, the spreading of a language or a religion of an ethnic or religious minority, and social reforms are examples of phenomena which can be modeled in this fashion.

To describe the dynamics of the system, let us focus on $\rho(t)$, the "popularity" of + opinions, defined as the average fraction of + nodes at time t. Thus, we consider $P(M, t)$, the probability of finding the network with M positive spins at time t, and

$$\rho(t) \equiv \sum_{M=0}^{N} \frac{M}{N} P(M, t), \tag{1}$$

where N is the total (fixed) number of nodes. Clearly, $\rho = 0$ or 1 correspond to a complete ordering of the system, while $\rho = 0.5$ characterizes the completely disordered state. Contrary to the voter model on regular lattices, the global magnetization ($m = 2\rho - 1$) is *not conserved* here, but the dynamics is still $\mathbb{Z}_2$ symmetric (i.e., invariant under the global inversion $\sigma_i \mapsto -\sigma_i$, $M/N \mapsto 1 - M/N$).

Since the spins on the nodes flip one at a time, the time evolution of $P(M,t)$ is a simple birth-death process for which we can write a master equation:

$$\partial_t P(M,t) = b_{M-1} P(M-1,t) + d_{M+1} P(M+1,t) - \left[b_M + d_M\right] P\left(M,t\right). \tag{2}$$

Here, b_M denotes the birth rate of a positive spin (i.e., the rate for flipping a negative to a positive spin), and d_M its death rate (flip rate from positive to negative). Both depend on M, the current number of positive spins in the system. Whether a positive spin will flip or not is determined by the number of positive and negative spins it is connected to. Since these connections are established randomly, with probabilities p and q, respectively, they are controlled by binomial distributions. For example, the probability that a positive spin is connected to exactly k of the other $M-1$ positive spins, is given by $B_{M-1,\,p}(k) \equiv \binom{M-1}{k} p^k (1-p)^{M-1-k}$. Writing a similar expression for the probability of this spin to be connected to k' negative spins, the death rate is given by

$$d_M = \frac{M}{N} \sum_{k=0}^{M-1} \sum_{k'=0}^{N-M} B_{M-1,\,p}(k)\, B_{N-M,\,q}(k')\, \Theta(k'-k).$$

The prefactor simply reflects the probability to find a positive spin among the N spins. The step function $\Theta(k'-k)$ expresses the fact that if the selected spin is connected to k positive and k' negative spins then, in the next time step, it will take the state $\mathrm{sgn}\,(k-k')$. Similarly, the birth rate is:

$$b_M = \frac{N-M}{N} \sum_{l=0}^{N-M-1} \sum_{l'=0}^{M} B_{N-M-1,p}(l) B_{M,q}(l') \Theta(l'-l).$$

For the special case $q = 1-p$ [21], the Θ-functions can be eliminated from b_M and d_M, due to the properties of the binomial distributions, to yield:

$$d_M = \frac{M}{N} \sum_{k=0}^{N-M-1} B_{N-1,p}(k), \tag{3}$$

$$b_M = \frac{N-M}{N} \sum_{l=0}^{M-1} B_{N-1,p}(l). \tag{4}$$

Before solving the master equation, we discuss its approximate solutions in the thermodynamic limit, in order to identify the different types of behavior and the parameter regimes where they occur. For $N \to \infty$, the binomial distribution $B_{N,p}(k)$ approaches a normal distribution with mean Np, so that d_M is given by the Gaussian error function multiplied by the prefactor M/N,

$$d_M \simeq \begin{cases} M/N, & \text{if } M < Np, \\ 0, & \text{if } M > Np, \end{cases} \tag{5}$$

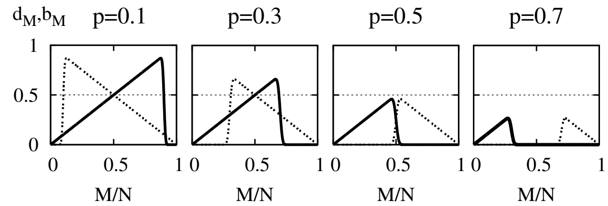

FIG. 1: The death d_M and birth d_M rates (continuous and dashed lines, respectively), for $N = 1000$ and different probabilities p. For fixed p, there are three different regimes of behavior bounded by Eq. (6).

apart from a region of width $\sqrt{Np(1-p)}$ around Np. Similarly, the birth rate b_M is described by the complementary error function. These forms of the transition rates (see Fig. 1), along with the probability p and the initial fraction of positive spins, M_0/N, determine the late-time properties of the model. The master equation controls the flow of M/N, as a function of time, leading to four distinct regimes, depending on the relative magnitudes of M_0/N and p. In a $(p, M_0/N)$ phase diagram, these different regimes are bounded by

$$\frac{M_0}{N} = p, \quad \text{and} \quad \frac{M_0}{N} = 1-p. \tag{6}$$

For $p < 0.5$ and $M_0/N < p$, we find that M/N stays below p at later time also. Indeed, in the approximation (5), we have a pure death process

$$\partial_t P(M,t) = (M+1)\, P(M+1,t) - M P\left(M,t\right), \tag{7}$$

which leads to the extinction of the positive population. The steady state, $\rho_\infty \equiv \lim_{t\to\infty} \rho(t) = 0$, is reached exponentially as $\rho(t) \sim \rho_0 \exp(-t/N)$. Similarly, if $p < 0.5$ and $M_0/N > 1-p$, we have a pure birth process, and the system relaxes exponentially to the state $\rho_\infty = 1$ on the same characteristic time scale as in the previous case (due to the $\mathbb{Z}_2$-symmetry). In the intermediate region $M_0/N \in [p, 1-p]$, the dynamics is described by:

$$N\partial_t P(M,t) = (M+1)P(M+1,t) + (N-M+1)P(M-1,t) - NP(M,t), \tag{8}$$

and the system reaches a disordered phase: $\rho_\infty = 0.5$. Again, the relaxation is exponential, with a characteristic time scale $N/2$.

For $p > 0.5$ the pure death and pure birth regimes are the same as for $p < 0.5$. A small minority ($M_0/N < 1-p$) will become extinct, a large majority ($M_0/N > p$) will win. However, a new feature appears in the interval $M_0/N \in [1-p, p]$, where the system seems to acquire infinite memory. Both the death and birth rates vanish in this region, so that $\partial_t P(M,t) = 0$ whence the fraction of positive spins remains frozen at its initial value, M_0/N.

Our analytic findings are tested by simulations [24] on a network with $N = 1,000$ nodes. The relaxation into the

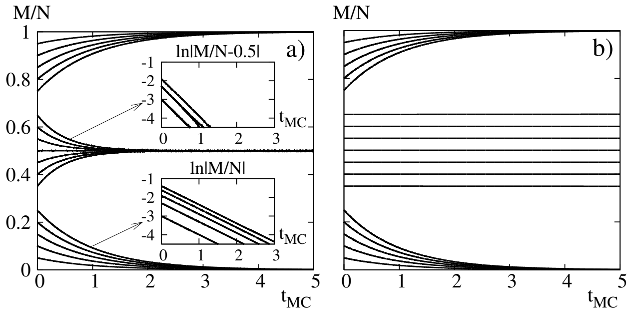

FIG. 2: Time evolution of the fraction M/N of positive spins for a network of $N = 1000$ nodes, for $p = 0.3$ (a) and $p = 0.7$ (b). The values are averaged over 1000 runs. Time is measured in Monte Carlo steps, $t_{\rm MC} = t_{\text{spin-flip}}/N$. The insets show the exponential relaxation to the final states.

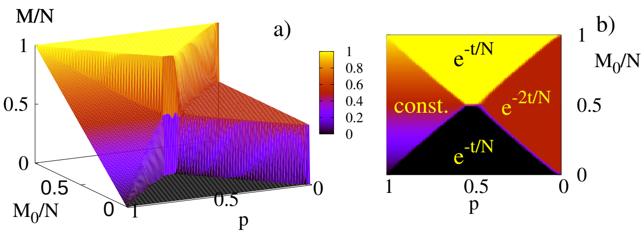

FIG. 3: The fraction of positive spins M/N after $t_{\rm MC} = 10$, as function of p and M_0/N for a network of $N = 1{,}000$ nodes in $3D$ (a), and $2D$ (b) displaying clearly the phase transitions.

four late-time states is displayed in Figs. 2(a) for $p < 0.5$ and (b) for $p > 0.5$. The possible outcomes of the voter dynamics, for all parameters p and initial fractions M_0/N of the positive population are summarized in Fig. 3.

To illustrate the picture further, Fig. 4 shows the outcome of the voter dynamics for two initial fractions of positive population: one starting from a minority $M_0/N < 0.5$, and one starting from a majority $M_0/N > 0.5$. For small p, the system reaches a disordered state, independent of M_0/N: the “open mindedness” of the population (reflected by a large probability $1-p$ to communicate with the opposite party) leads to an equal distribution of opinions. In contrast, an “inflexible attitude” (characterized by a large probability p of linking up with similar opinions) leads to an unchanging distribution of opinions. For intermediate values of p, the system reaches a completely ordered state: all voters reach the same opinion, namely that of the initial majority.

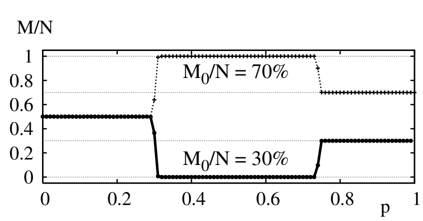

FIG. 4: Outcome of the voter dynamics as function of p. Cross sections of Fig. 3 taken along $M_0/N = 0.3$ and $M_0/N = 0.7$.

In conclusion, in the thermodynamic limit, the voter dynamics has four possible outcomes: a perfect balance of opinions, a static situation, or consensus ($\rho = 0, 1$). In this last section, we discuss how these findings are modified in *finite* systems. The two completely ordered states are absorbing states, thus they will be reached from the other two (metastable) states. Two interesting questions remain: First, in *which* of the two absorbing states will each metastable state arrive, and second, how do the relaxation times depend on system size?

To answer the first question, we write Eq. (2) in a matrix representation, $\partial_t\,|v(t)\rangle = \mathbb{L}\,|v(t)\rangle$, where $|v(t)\rangle$ is the $(N+1)$-dimensional column vector with components $P(M,t)$, $M = 0, 1, ..., N$, and the time evolution operator $\mathbb{L}$ can be read off from Eq. (2). The steady states of the system are the eigenvectors of $\mathbb{L}$ with zero eigenvalues: $|0_0\rangle \equiv \left(1,\ 0,\ ...,\ 0\right)^{\mathsf{T}}$ and $|0_N\rangle \equiv \left(0,\ 0,\ ...,\ 1\right)^{\mathsf{T}}$. To find their adjoints, it is convenient to study the symmetric/antisymmetric states: $|0_\pm\rangle \equiv \big(|0_0\rangle \pm |0_N\rangle\big)/2$. Imposing orthonormality $\langle 0_\pm|0_\pm\rangle = \delta_\pm$, we obtain from $\langle 0_\pm|\,\mathbb{L} = 0$ the right eigenvectors $\langle 0_+| \equiv \big(1,\ 1,\ ...,\ 1\big)$ and $\langle 0_-| \equiv \big(1,\ x_1,\ x_2,\ ...,\ -x_2,\ -x_1,\ -1\big)$, with

$$x_j \equiv \frac{\big(((r_{j+1}+1)r_{j+2}+1)r_{j+3}+...+1\big)r_{n-1}+1}{\big(((r_1+1)r_2+1)r_3+...+1\big)r_{n-1}+1},$$

where $r_j \equiv b_j/d_j$. Thus we can compute explicitly the final state $|\psi_\infty\rangle \equiv \lim_{t\to\infty}|\psi_t\rangle$. Given an initial state $|\psi_0\rangle$, the solution to Eq. (2) is $|\psi_t\rangle = \sum_\mu e^{-\lambda_\mu t}|\mu\rangle\langle\mu|\psi_0\rangle$, where $\langle\mu|$ and $|\mu\rangle$ are the left and right eigenvectors of $\mathbb{L}$ corresponding to eigenvalues λ_μ. Expecting no other zero λ_μ's, we have $|\psi_\infty\rangle = |0_+\rangle\langle 0_+|\psi_0\rangle + |0_-\rangle\langle 0_-|\psi_0\rangle$. For example, if the initial state is a population with $M_0 < N/2$ positive spins, the final state is

$$|\psi_\infty\rangle = \frac{1}{2}\left(1{+}x_{M_0},\ 0,\ ...,\ 0,\ 1{-}x_{M_0}\right)^{\mathsf{T}}. \qquad (9)$$

This result shows that, indeed, in the $t \to \infty$ limit, the system arrives in one of its two absorbing states. Moreover, it provides the *relative probabilities* with which either will be reached, as a function of the initial M_0. Further details will be published elsewhere [21].

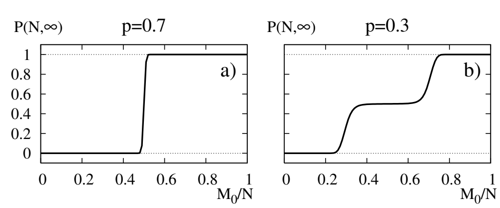

FIG. 5: The probability $P(N,\infty) = (1-x_M)/2$, to reach the state with $M = N$ positive spins at infinite times: $P(N,\infty) = 0$ implies certain extinction of the positive population, while $P(N,\infty) = 1$ represents a purely positive population.

These findings are confirmed in Fig. 5, obtained by direct iteration of the master equation. For $p > 0.5$, the

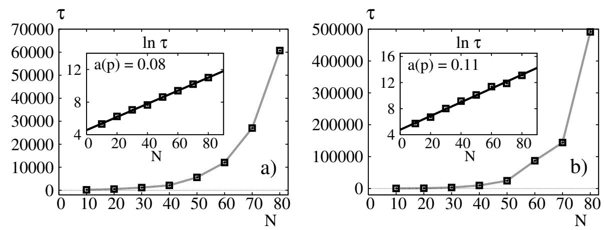

FIG. 6: Time τ needed to reach consensus as function of the system size for $p = 0.3$, $M_0/N = 0.4$ (a), and $p = 0.8$, $M_0/N = 0.4$ (b). The insets show the exponential dependence on N.

dynamics reduces to a simple majority rule: The final state is completely ordered, following the opinion of the initial majority. For $p < 0.5$, a small minority disappears, while a big majority wins the competition, just as in the case of infinite system size. A novel effect occurs for *intermediate* values of the initial positive population. In *finite* systems, the fully disordered (metastable) state has a finite lifetime, during which the system forgets its initial condition. Then, after a very long time, it *randomly* falls into one of its absorbing states. To rephrase, the initial positive population is equally likely to become extinct or to take over the whole system. In this regime, the final outcome of the voter dynamics is completely random.

Finally, we explore numerically the relaxation times into the absorbing states in Fig. 6. The lifetime τ of the metastable states increases with the system size N, as $\tau \sim e^{a(p)N}$, with a p-dependent coefficient $a(p)$. An estimate of the time to consensus for a network of $N = 1000$ voters with $p = 0.3$ shows that it may take as many as 10^{36} spin flips to reach one of the absorbing states. In much larger systems, consensus is practically impossible.

In conclusion, the best strategy for a minority group is to establish many contacts with its opponents. In this way, it can convince half of them and keep this balance for a long time. If the same group is less open for discussions, it cannot overcome the majority, but at least it will not disappear. It is tempting to speculate how these results might be applied to real social systems. Will two-party systems, once formed, persist for long times? Will bilingual regions remain bilingual? Will relatively isolated parties continue to receive the same, almost constant percentage of the vote? Will closed religious communities continue to exist without gaining or losing members?

These particular results are obtained from a simple adaptive model in which certain important social factors are neglected (e.g., spatial and age structures, a spectrum of opinions, etc.). More precise statements will certainly need additional assumptions on the character of the network or opinion dynamics. Nevertheless, even this simple model can give a better understanding of empirical data not yet explained. For example, a study of the number of languages in the Solomon Islands [23] found that small islands (less than 100 square miles) have a single language, but above this size the number of languages increases. The finite size effect pointed out by our model can be a possible explanation of this phenomenon.

The importance of the model presented here stems from the fact that it is mathematically simple, exactly solvable, and easily generalized to more complex situations (for $q \neq 1 - p$ see [21], for epidemics networks see [22]). Our model provides a *method* how to describe analytically the adaptive nature of the interpersonal relations, and by this, it can serve as a "baseline model" which captures the key characteristics of social systems, namely, having disordered networks of agents and dynamically changing connections between them.

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